Chapter 2 : Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Lecture 9 : Exercises & Correction

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Exercice 1

A.

Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$
b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$
c) $\{\emptyset\} \in \{\emptyset\}$
d) $\{\emptyset\} \in \{\{\emptyset\}\}$
e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

B.

Determine whether each of these statements is true or false.

a) $x \in \{x\}$
b) $\{x\} \subseteq \{x\}$
c) $\{x\} \in \{x\}$
d) $\{x\} \in \{\{x\}\}$
e) $\emptyset \subseteq \{x\}$
f) $\emptyset \in \{x\}$

Solution Exercise 1-A

(a) Given: $\emptyset \in \{\emptyset\}$

$\{\emptyset\}$ represent the set containing only the empty set and thus the empty set is an element of $\{\emptyset\}$, which means that the given statement is true.

(b) Given: $\emptyset \in \{\emptyset, \{\emptyset\}\}$

The given statement means that the empty set is an element of the set containing the element $\emptyset$ and the subset $\{\emptyset\}$.

We then note that $\emptyset$ is an element of the set $\{\emptyset, \{\emptyset\}\}$, which means that the given statement is true.

(c) Given: $\{\emptyset\} \in \{\emptyset\}$

The given statement means that the set $\{\emptyset\}$ is an element of the set containing the empty set.

The set containing the empty set does not contain any sets which contain elements. The set containing the empty set is a set which contains elements, thus the given statement is false.

(d) Given: $\{\emptyset\} \in \{\{\emptyset\}\}$

The given statement means that the set $\{\emptyset\}$ is an element of the set containing the set $\{\emptyset\}$.

Since $\{\emptyset\}$ is the only set contained in the set $\{\{\emptyset\}\}$, the given statement is true.

(e) Given: $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

The given statement means that the set $\{\emptyset\}$ is a subset of the set containing element $\emptyset$ and set $\{\emptyset\}$.

The set $\{\emptyset\}$ contains only the element $\emptyset$. Since $\emptyset$ is also an element in $\{\emptyset, \{\emptyset\}\}$, the given statement is true.

(f) Given: $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

The given statement means that the set $\{\{\emptyset\}\}$ is a subset of the set containing elements $\emptyset$ and $\{\emptyset\}$.

The set $\{\{\emptyset\}\}$ contains only the element $\{\emptyset\}$. Since $\{\emptyset\}$ is also an element in $\{\emptyset, \{\emptyset\}\}$, the given statement is true.

(g) Given: $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

The given statement means that the set $\{\{\emptyset\}\}$ is a subset of the set containing the element $\{\emptyset\}$.

The set $\{\{\emptyset\}\}$ contains only the element $\{\emptyset\}$ and since $\{\{\emptyset\}, \{\emptyset\}\}$ contains only the element $\{\emptyset\}$ as well, the two sets are the same set. The given statement is then false as $\{\{\emptyset\}\} = \{\{\emptyset\}, \{\emptyset\}\}$.

Note: the statement $\{\{\emptyset\}\} \subseteq \{\{\emptyset\}, \{\emptyset\}\}$ would be true (as the sets are equal).

Solution Exercise 1-B

(a) Given: $x \in \{x\}$

The given statement means that x is an element of the set $\{x\}$.

The set $\{x\}$ contains only the element x and thus the given statement is true.

(b) Given: $\{x\} \subseteq \{x\}$

The given statement means that the set $\{x\}$ is an inclusive subset of $\{x\}$.

Every subset is an inclusive subset of itself, which means that the given statement is true.

(c) Given: $\{x\} \in \{x\}$

The given statement means that the set $\{x\}$ is an element of the set $\{x\}$.

The set $\{x\}$ does not contain any sets as elements, thus the given statement is false.

(d) Given: $\{x\} \in \{\{x\}\}$

The given statement means that $\{x\}$ is an element of the set $\{\{x\}\}$.

The set $\{\{x\}\}$ contains only the element $\{x\}$ and thus the given statement is true.

(e) Given: $\emptyset \subseteq \{x\}$

The given statement means that the set $\emptyset$ is an inclusive subset of the set $\{x\}$.

The empty set is a subset of every set and thus the given statement is true.

(f) Given: $\emptyset \in \{x\}$

The given statement means that the set $\emptyset$ is an element of the set $\{x\}$.

The set $\{x\}$ contains only the element x and thus does not contain the element $\emptyset$. This then means that the given statement is false.

Exercice 2

a)

Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

b)

Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

c)

Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

Solution Exercise 2

To proof: $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$

PROOF

$(i) \Rightarrow (ii)$

Let us assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Let x be an element of A.

$$x \in A$$

If x is an element in a set S, then the set containing only that element x is a set in the power set of S.

$$\{x\} \in \mathcal{P}(A)$$

Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$:

$$\{x\} \in \mathcal{P}(B)$$

If the set containing only that element x is a set in the power set of S, then the element x has to be an element of the set S.

$$x \in B$$

We thus have derived that every element x in A also has to be in B. By the definition of a subset, we then know that A is a subset of B.

$$A \subseteq B$$

$(ii) \Rightarrow (i)$
Let us assume $A \subseteq B$.

If $\mathcal{P}(A)$ contains only the empty set $\emptyset$, then the proof is trivial as any power set contains the emptyset $\emptyset$.
If $\mathcal{P}(A)$ does not contain only the empty set $\emptyset$, then there exists a set of the form $\{x\}$ in $\mathcal{P}(A)$.

Let $\{x\}$ be an element of $\mathcal{P}(A)$.

$$\{x\} \in \mathcal{P}(A)$$

If the set containing only an element x is a set in the power set of S, then the element x has to be an element of the set S.

$$x \in A$$

Since $A \subseteq B$:

$$x \in B$$

If x is an element in a set S, then the set containing only that element x is a set in the power set of S.

$$\{x\} \in \mathcal{P}(B)$$

We thus have derived that every element x in $\mathcal{P}(A)$ also has to be in $\mathcal{P}(B)$. By the definition of a subset, we then know that $\mathcal{P}(A)$ is a subset of $\mathcal{P}(B)$.

$$\mathcal{P}(A) \subseteq \mathcal{P}(B)$$

Given:

$$A \subseteq C$$

$$B \subseteq D$$

To proof: $A \times B \subseteq C \times D$

PROOF

Let (a, b) be an element of $A \times B$.

$$(a, b) \in A \times B$$

By the definition of the Cartesian product:

$$a \in A$$

$$b \in B$$

If X is a subset of Y, then every element of X is also an element of Y. By the given statements $A \subseteq C$ and $B \subseteq D$, we then obtain:

$$a \in C$$

$$b \in D$$

By the definition of the Cartesian product:

$$(a, b) \in C \times D$$

We thus have derived that every element (a, b) in $A \times B$ also has to be in $C \times D$. By the definition of a subset, we then know that $A \times B$ is a subset of $C \times D$.

$$A \times B \subseteq C \times D$$

Exercice 3

A.

What is the cardinality of each of these sets?

a) $\{a\}$

b) $\{\{a\}\}$

c) $\{a, \{a\}\}$

d) $\{a, \{a\}, \{a, \{a\}\}\}$

B.

How many elements does each of these sets have where a and b are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\emptyset))$

C.

Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) $A \times B$.

b) $B \times A$.

Solution Exercise 3-A

a.) The set contains just 1 element so cardinality = 1
b.) The set contains just 1 element so cardinality = 1
c.) The set contains 2 elements, so cardinality = 2
d.) The set contains 3 elements, so cardinality = 3

Solution Exercise 3-B

a. 8
b. 16
c. 2

Solution Exercise 3-C

PART A: $\{(a, y), (a, z), (b, y), (b, z), (c, y), (c, z), (d, y), (d, z)\}$
PART B: $\{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}$

Exercice 4

A.

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. **b)** $A \cap B$.

c) $A - B$. **d)** $B - A$.

B.

Show that

a) $A - \emptyset = A$. **b)** $\emptyset - A = \emptyset$.

Solution Exercise 4-A

PART A: $\{0, 1, 2, 3, 4, 5, 6\}$

PART B: $\{3\}$

PART C: $\{1, 2, 4, 5\}$

PART D: $\{0, 6\}$

Solution Exercise 4-B

(a) Given: U is the universal set

To proof: $A - \emptyset = A$

PROOF

$$A - \emptyset = \{x | x \in A - \emptyset\}$$

Using the definition of the difference, an element of $A - \emptyset$ is an element that is in A but not in $\emptyset$.

$$= \{x | x \in A \wedge x \notin \emptyset\}$$

The empty set does not contain any elements, thus the statement $x \notin \emptyset$ is always true.

$$= \{x | x \in A \wedge \mathbf{T}\}$$

Use the identity law for propositions:

$$= \{x | x \in A\}$$

$$= A$$

We have then shown $A - \emptyset = A$.

(b) Given: U is the universal set

To proof: $\emptyset - A = \emptyset$

PROOF

$$\emptyset - A = \{x|x \in \emptyset - A\}$$

Using the definition of the difference, an element of $\emptyset - A$ is an element that is in $\emptyset$ but not in A.

$$= \{x|x \in \emptyset \wedge x \notin A\}$$

The emptyset does not contain any elements, thus the statement $x \in \emptyset$ is always false:

$$= \{x|\mathbf{F} \wedge x \notin A\}$$

Use the domination law for propositions:

$$= \{x|\mathbf{F}\}$$

The emptyset does not contain any elements, thus the statement $x \in \emptyset$ is always false:

$$= \{x|x \in \emptyset\}$$

$$= \emptyset$$

We have then shown $\emptyset - A = \emptyset$.

Exercice 5

A.

Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.

B.

Prove the identity laws in Table 1 by showing that

a) $A \cup \emptyset = A$. **b)** $A \cap U = A$.

C.

Prove the domination laws in Table 1 by showing that

a) $A \cup U = U$. **b)** $A \cap \emptyset = \emptyset$.

D.

Prove the complement laws in Table 1 by showing that

a) $A \cup \overline{A} = U$. **b)** $A \cap \overline{A} = \emptyset$.

Solution Exercise 5-A

Given: U is the universal set

To proof: $\overline{\overline{A}} = A$

PROOF

Using the definition of the complement, an element of $\overline{\overline{A}}$ is an element that is not in $\overline{A}$.

$$\overline{\overline{A}} = \{x | x \notin \overline{A}\}$$

$$= \{x | \neg(x \in \overline{A})\}$$

Using the definition of the complement, an element of $\overline{A}$ is an element that is not in A.

$$= \{x | \neg(x \notin A)\}$$

$$= \{x | \neg(\neg(x \in A))\}$$

Using the double negation law:

$$= \{x | x \in A\}$$

$$= A$$

We have then shown $\overline{\overline{A}} = A$.

Solution Exercise 5-B

a) $A \cup \phi = A$

$$\begin{aligned} \text{a) L.H.S} &= A \cup \phi = \{x : x \in A \cup \phi\} \\ &= \{x : x \in A \text{ or } x \in \phi\} \\ &= \{x : x \in A\} \\ &= A \\ &= \text{R.H.S} \end{aligned}$$

b) $A \cap U = A$

$$\begin{aligned} \text{b) L.H.S} &= A \cup U \\ &= \{x : x \in A \cap U\} \\ &= \{x \in A \text{ and } x \in U\} \\ &= \{x : x \in A\} \\ &= A \\ &= \text{R.H.S} \end{aligned}$$

Solution Exercise 5-C

(a) Given: U is the universal set

To proof: $A \cup U = U$

PROOF

$$A \cup U = \{x | x \in A \cup U\}$$

Using the definition of the union, an element of $A \cup U$ is an element that is in A or in U.

$$= \{x | x \in A \vee x \in U\}$$

Every element is contained in the universal set U, thus the statement $x \in U$ is always true.

$$= \{x | x \in A \vee \mathbf{T}\}$$

Use the domination law for propositions:

$$= \{x | \mathbf{T}\}$$

We derived that $x \in U$ was a true statement, thus we can replace $\mathbf{T}$ again by $x \in U$:

$$\{x | x \in U\}$$

$$= U$$

We have then shown $A \cup U = U$.

(b) Given: U is the universal set

To proof: $A \cap \emptyset = \emptyset$

PROOF

$$A \cap \emptyset = \{x | x \in A \cap \emptyset\}$$

Using the definition of the intersection, an element of $A \cap \emptyset$ is an element that is in A and in U.

$$= \{x | x \in A \wedge x \in \emptyset\}$$

The empty set does not contain any elements and thus it is not possible that x is an element of $\emptyset$.

$$= \{x | x \in A \wedge \mathbf{F}\}$$

Use the domination law for propositions:

$$= \{x | \mathbf{F}\}$$

We derived that $x \in \emptyset$ was a false statement, thus we can replace $\mathbf{F}$ again by $x \in \emptyset$:

$$= \{x | x \in \emptyset\}$$

$$= \emptyset$$

We have then shown $A \cap \emptyset = \emptyset$.

Solution Exercise 5-D

(a) Given: U is the universal set

To proof: $A \cup \overline{A} = U$

PROOF

$$A \cup \overline{A} = \{x | x \in A \cup \overline{A}\}$$

Using the definition of the union, an element of $A \cup \overline{A}$ is an element that is in A or in $\overline{A}$.

$$= \{x | x \in A \vee x \in \overline{A}\}$$

An element is in $\overline{A}$ when the element is not in A:

$$= \{x | x \in A \vee x \notin A\}$$
$$= \{x | x \in A \vee \neg(x \in A)\}$$

Use the negation law for propositions:

$$= \{x | \mathbf{T}\}$$

Every element is in the universal set, thus $x \in U$ is a true statement:

$$= \{x | x \in U\}$$
$$= U$$

We have then shown $A \cup \overline{A} = U$.

(b) Given: U is the universal set

To proof: $A \cap \overline{A} = \emptyset$

PROOF

$$A \cap \overline{A} = \{x | x \in A \cap \overline{A}\}$$

Using the definition of the intersection, an element of $A \cap \overline{A}$ is an element that is in A and in $\overline{A}$.

$$= \{x | x \in A \wedge x \in \overline{A}\}$$

An element is in $\overline{A}$ when the element is not in A:

$$= \{x | x \in A \wedge x \notin A\}$$

$$= \{x | x \in A \wedge \neg(x \in A)\}$$

Use the negation law for propositions:

$$= \{x | \mathbf{F}\}$$

The empty set does not contain any elements, thus $x \in \emptyset$ is a false statement.

$$= \{x | x \in \emptyset\}$$

$$= \emptyset$$

We have then shown $A \cap \overline{A} = \emptyset$.